Arrow's Impossibility Theorem

Lecture 11
Section 1.6

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Hampden-Sydney College

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1. The Majority Criterion
2. The Condorcet Criterion
3. The Monotonicity Criterion
4. The Independence-of-Irrelevant-Alternatives Criterion
5. Arrow's Impossibility Theorem
6. Assignment

Outline

1. The Majority Criterion
2. The Condorcet Criterion
3. The Monotonicity Criterion
4. The Independence-of-Irrelevant-Alternatives Criterion
5. Arrow's Impossibility Theorem
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The Majority Criterion

Definition (Majority Criterion)

The Majority Criterion says that if a candidate has a majority of first-place votes, then that candidate should be the winner. (There may or may not be a majority candidate.)

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- The Majority Criterion says only which candidate *should* be the winner, not which candidate *will* be the winner.
- The Borda count method and Coombs' method may violate the Majority Criterion.

The Majority Criterion

Example (The Majority Criterion – Borda Count Method)

No. of Votes	8	6	5	3
1st	A	A	B	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	D	C	A	D

- Does any candidate have a majority?

The Majority Criterion

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No. of Votes	8	6	5	3
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- Then who “should” win?

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- Does any candidate have a majority? Yes
- Then who "should" win? A
- Who wins by the Borda Count Method? B

The Majority Criterion

Example (The Majority Criterion – Coomb's Method)

No. of Votes	8	6	5	2
1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

- Does any candidate have a majority?

The Majority Criterion

Example (The Majority Criterion – Coomb's Method)

No. of Votes	8	6	5	2
1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

- Does any candidate have a majority? Yes

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Example (The Majority Criterion – Coomb's Method)

No. of Votes	8	6	5	2
1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

- Does any candidate have a majority? Yes
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No. of Votes	8	6	5	2
1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

- Does any candidate have a majority? Yes
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1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

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- Then who "should" win? A
- Who wins by Coomb's Method?

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1st	D	A	A	B
2nd	B	B	C	C
3rd	C	D	D	A
4th	A	C	B	D

- Does any candidate have a majority? Yes
- Then who "should" win? A
- Who wins by Coomb's Method? D

The Majority Criterion

- The Borda count method and Coombs' method *may* violate the Majority Criterion.
- The plurality method obviously does not violate the Majority Criterion.
- The other methods (I'm pretty sure) do not violate it.

Outline

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The Condorcet Criterion

Definition (Condorcet Winner)

The Condorcet winner is a candidate who beat *every other candidate* in pairwise comparisons. (There may or may not be a Condorcet winner.)

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The Condorcet Criterion says that if there is a Condorcet winner, then that candidate should be the winner (by whatever method used).

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- If there is no Condorcet winner, then the Condorcet Criterion cannot be violated.
- The Condorcet Criterion says only which candidate *should* be the winner, not which candidate *will* be the winner.

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Example (The Condorcet Criterion – Borda Count Method)

No. of Votes	8	6	5	3
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- Is there a Condorcet winner?

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1st	A	A	B	B
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- Is there a Condorcet winner? Yes, A

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3rd	D	C	A	D

- Is there a Condorcet winner? Yes, A
- Then A “should” win.

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Example (The Condorcet Criterion – Borda Count Method)

No. of Votes	8	6	5	3
1st	A	A	B	B
2nd	B	B	C	C
3rd	C	D	D	A
3rd	D	C	A	D

- Is there a Condorcet winner? Yes, A
- Then A “should” win.
- Who wins by the Borda Count Method?

The Condorcet Criterion

Example (The Condorcet Criterion – Borda Count Method)

No. of Votes	8	6	5	3
1st	A	A	B	B
2nd	B	B	C	C
3rd	C	D	D	A
3rd	D	C	A	D

- Is there a Condorcet winner? Yes, A
- Then A "should" win.
- Who wins by the Borda Count Method? B

The Condorcet Criterion

- The plurality method, Borda count method, plurality-with-elimination method, and Coombs' method may violate the Condorcet Criterion.
- The method of pairwise comparisons obviously does not violate it.

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- The Monotonicity Criterion says only that the winning candidate *should* still be the winner, not that the winning candidate *will* still be the winner.

The Monotonicity Criterion

Example (The Monotonicity Criterion – Plurality-with-Elimination Method)

	10	8	7	4
1st	B	A	C	C
2nd	C	B	A	B
3rd	A	C	B	A

- Who is the winner by the Plurality-with-Elimination Method?

The Monotonicity Criterion

Example (The Monotonicity Criterion – Plurality-with-Elimination Method)

	10	8	7	4
1st	B	A	C	C
2nd	C	B	A	B
3rd	A	C	B	A

- Who is the winner by the Plurality-with-Elimination Method? B

The Monotonicity Criterion

Example (The Monotonicity Criterion – Plurality-with-Elimination Method)

	10	8	7	4
1st	B	A	C	C
2nd	C	B	A	B
3rd	A	C	B	A

- Who is the winner by the Plurality-with-Elimination Method? B
- Then B "should" win even if the last 4 voters decide to rank B over C.

The Monotonicity Criterion

Example (The Monotonicity Criterion – Plurality-with-Elimination Method)

	10	8	7	4
1st	B	A	C	B
2nd	C	B	A	C
3rd	A	C	B	A

- Who is the winner by the Plurality-with-Elimination Method? B
- Then B "should" win even if the last 4 voters decide to rank B over C.
- Suppose that the last four voters decide to rank B over C. Now who is the winner?

The Monotonicity Criterion

Example (The Monotonicity Criterion – Plurality-with-Elimination Method)

	10	8	7	4
1st	B	A	C	B
2nd	C	B	A	C
3rd	A	C	B	A

- Who is the winner by the Plurality-with-Elimination Method? B
- Then B "should" win even if the last 4 voters decide to rank B over C.
- Suppose that the last four voters decide to rank B over C. Now who is the winner? A

The Monotonicity Criterion

- The plurality-with-elimination Method may violate the Monotonicity Criterion.
- The other methods (I'm pretty sure) do not violate it.

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The Independence-of-Irrelevant-Alternatives Criterion

Definition (Independence-of-Irrelevant-Alternatives Criterion)

The independence-of-irrelevant-alternatives criterion (IIA) says that if candidate X is the winner, then X would still be the winner if one or more of the losing candidates had not been in the race.

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The independence-of-irrelevant-alternatives criterion (IIA) says that if candidate X is the winner, then X would still be the winner if one or more of the losing candidates had not been in the race.

- The IIA Criterion says only that the winning candidate *should still be* the winner, not that the winning candidate *still is* the winner.

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The independence-of-irrelevant-alternatives criterion (IIA) says that if candidate X is the winner, then X would still be the winner if one or more of the losing candidates had not been in the race.

- The IIA Criterion says only that the winning candidate *should still be* the winner, not that the winning candidate *still is* the winner.
- Check out the story of Sidney Morgenbesser.

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	A	B
2nd	B	B	A

- The group of 5 is offered a choice between Apple and Blueberry

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	A	B
2nd	B	B	A

- The group of 5 is offered a choice between Apple and Blueberry
- By “elimination” which pie do they choose?

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	A	B
2nd	B	B	A

- The group of 5 is offered a choice between Apple and Blueberry
- By “elimination” which pie do they choose? Apple

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- The waitress comes back and includes Cherry as a third option.

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- The waitress comes back and includes Cherry as a third option.
- Now which pie do they choose, by elimination?

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Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- The waitress comes back and includes Cherry as a third option.
- Now which pie do they choose, by elimination? Blueberry

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.
- Who is the winner by the Plurality-with-Elimination Method?

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.
- Who is the winner by the Plurality-with-Elimination Method? B

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.
- Who is the winner by the Plurality-with-Elimination Method? B
- Suppose that candidate C drops out. Now who is the winner?

The Independence-of-Irrelevant-Alternatives Criterion

Example (The Independence-of-Irrelevant-Alternatives Criterion – Plurality-with-Elimination Method)

	1	2	2
1st	A	C	B
2nd	B	A	A
3rd	C	B	C

- Independence of Irrelevant Alternatives runs that example in reverse.
- Who is the winner by the Plurality-with-Elimination Method? B
- Suppose that candidate C drops out. Now who is the winner? A

The Independence-of-Irrelevant-Alternatives Criterion

- The plurality method and the plurality-with-elimination method may violate the IIA Criterion.
- The other methods (I'm pretty sure) do not violate it.

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Arrow's Impossibility Theorem

Theorem (Arrow's Impossibility Theorem)

If there are at least 3 candidates, then there is no voting method that cannot violate any of the four desired properties (Majority, Condorcet, Monotonicity, Independence of Irrelevant Alternatives).

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- Chapter 1 Exercises 51, 52, 53, 54, 55, 56.